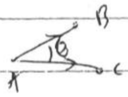
1401 Mathematical Methods 1
May 2009 Solutions

1. a) i) $\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\theta$ with $|\underline{a}|$ the modulus of $\underline{a}$ & θ the angle between $\underline{a}$ & $\underline{b}$

ii) $\underline{a}\wedge\underline{b} = |\underline{a}||\underline{b}|\sin\theta\,\hat{\underline{c}}$ with $\hat{\underline{c}}$ normal to $\underline{a}$ & $\underline{b}$ & such that $\underline{a}, \underline{b}, \underline{a}\wedge\underline{b}$ form a right-handed set.

iii)

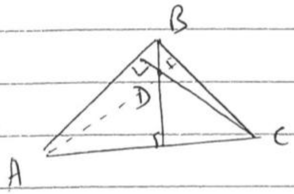


Area is $\frac{1}{2} AC\, AB \sin\theta = \frac{1}{2}|\vec{AC}\wedge\vec{AB}|$ (seen)

$= \frac{1}{2}|(\underline{c}-\underline{a})\wedge(\underline{b}-\underline{a})| = \frac{1}{2}|\underline{c}\wedge\underline{b} - \underline{c}\wedge\underline{a} - \underline{a}\wedge\underline{b} + \underbrace{\underline{a}\wedge\underline{a}}_{0}|$

$= \frac{1}{2}|\underline{a}\wedge\underline{b} + \underline{b}\wedge\underline{c} + \underline{c}\wedge\underline{a}|$

b)



$BD \perp AC$ so $\vec{AC}\cdot(\vec{DA} + \vec{AB}) = 0$

$DC \perp AB$ so $\vec{AB}\cdot(\vec{DA} + \vec{AC}) = 0$

subtract these $\Rightarrow (\vec{AC} - \vec{AB})\cdot\vec{DA} = 0$ ie $\vec{BC}\cdot\vec{DA} =$

Hence $AD \perp BC$ & result

2. a) i) $z^3 = 2 + i = \sqrt{5}\, e^{i(\theta + 2n\pi)}$, $\theta = \tan^{-1}\frac{1}{2}$. If $z = re^{i\varphi}$, then $r^3 = \sqrt{5}$ & $3\varphi = \theta + 2n\pi$. Hence $z_n = 5^{1/6}\left(\cos\left(\frac{\theta}{3} + \frac{2n\pi}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{2n\pi}{3}\right)\right)$ $n = 0, 1, 2$. (4)

ii) $z^2 + \frac{2}{i}z + \frac{1}{i}(1-i) = z^2 - 2i + (-1-i) = 0$, $z = \left(2i \pm \sqrt{-4 + 4 + 4i}\right)/2$

$z = i \pm \sqrt{i} = \pm\frac{1}{\sqrt{2}} + i\left(1 \pm \frac{1}{\sqrt{2}}\right)$

iii) $e^z = 1 + i \Rightarrow e^{x+iy} = e^x e^{iy} = \sqrt{2}\, e^{i(\pi/4 + 2n\pi)} \Rightarrow e^x = \sqrt{2}, y = \pi/4 + 2$

$\Rightarrow z = \frac{1}{2}\ln 2 + i(\pi/4 + 2n\pi)$ (5) (unseen)

b) i) $\omega_n = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ & so $\omega_n^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n$

$= \cos 2\pi + i\sin 2\pi = 1$ from DMTh

$S_n = 1 + \omega_n + \omega_n^2 + \dots \omega_n^{n-1}$, then $\omega_n S_n = \omega_n + \omega_n^2 + \dots + \cancel{\omega_n^n}^{1}$ so $(1-\omega_n)S_n = 0$ & as $\omega_n \neq 0$, $S_n = 0$.

ii) $(x + \omega_3 y + \omega_3^2 z)(x + \omega_3^2 y + \omega_3 z) = x^2 + \cancel{\omega_3^3}^{1} y^2 + \cancel{\omega_3^3}^{1} z^2 + xy(\underbrace{\omega_3^2 + \omega_3}_{-1}) + xz(\underbrace{\omega_3 + \omega_3^2}_{-1}) + zy(\underbrace{\omega_3^2 + \omega_3^4}_{-1})$

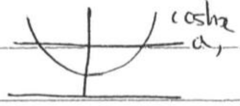
But $\omega_3^3 = 1$ & $1 + \omega_3 + \omega_3^2 = 0$ so $\omega_3 + \omega_3^2 = -1$. Also, multiplying by ω_3^2, $\omega_3^2 + \omega_3^3 + \omega_3^4 = 0$ & $\omega_3^2 + \omega_3^4 = -\omega_3^3 = -1$

$\Rightarrow$ result

3. a) i) $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

ii) If $\cosh x = a$, then $e^x + e^{-x} = 2a \Rightarrow e^{2x} - 2ae^x + 1 = 0 \Rightarrow e^x = (2a \pm \sqrt{4a^2-4}$

$\Rightarrow x = \ln(a \pm \sqrt{a^2-1})$. Sum is $\ln(a + \sqrt{a^2-1}) + \ln(a - \sqrt{a^2-1}) =$

$\ln(a^2 - (\sqrt{a^2-1})^2) = \ln 1 = 0$. 

, cosh is even.

iii) $\int \frac{dx}{\cosh x} = \int \frac{2\,dx}{e^x + e^{-x}} = \int \frac{2e^x}{e^{2x}+1}\,dx = \int \frac{2\,du}{u^2+1}$, $u = e^x$

$= 2\tan^{-1} u + C = 2\tan^{-1} e^x + C$,

b) $\frac{d}{dx} \tan^{-1}\left(\frac{a+x}{1-ax}\right) = \frac{1}{1+\left(\frac{a+x}{1-ax}\right)^2} \left\{ \frac{1}{1-ax} - \frac{(-a)(a+x)}{(1-ax)^2} \right\}$

$= \frac{1}{(1-\cancel{2ax}+a^2x^2)+(a^2+\cancel{2ax}+x^2)} \left\{ 1 - \cancel{ax} + a^2 + \cancel{ax} \right\} = \frac{(1+a^2)}{(1+a^2)(1+x^2)} = \frac{1}{1+x^2}$

This is $\frac{d}{dx} \tan^{-1} x$.

Note $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ & if $\tan A = a$, $\tan B = x$ we have

$\frac{d}{dx} \tan^{-1} \tan(A+B) = \frac{d(A+B)}{dx} = \frac{d}{dx}\left(\tan^{-1} a + \tan^{-1} x\right) = \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$, as a is constant

(as expected)

4. a) i) $\int (3x-4)^{3/2}\,dx = (3x-4)^{5/2}\cdot\frac{2}{5}\cdot\frac{1}{3} = \frac{2}{15}(3x-4)^{5/2} + \text{Const}$.

ii) $\int_0^{\pi/3} \frac{\sin x}{\cos^4 x}\,dx = \left[\frac{1}{3}\cos^{-3}x\right]_0^{\pi/3} = \frac{1}{3}\left(\left(\frac{1}{2}\right)^{-3} - 1\right) = \underline{7/3}$.

iii) $\int_0^{\pi/4} \tan^n x \sec^2 x\,dx = \left[\frac{\tan^{n+1}x}{n+1}\right]_0^{\pi/4} = \underline{\underline{1}}/(n+1)$

b) $\int \frac{dx}{(x-p)\sqrt{(x-p)(x-q)}} = \int \frac{-1/u^2\,du}{1/u\,\sqrt{1/u\,(p-q+1/u)}} = \int \frac{-du}{\sqrt{1+(p-q)u}} = \frac{2}{q-p}\sqrt{1+(p-q)u} + C$

$x = p + 1/u$ $\quad = \frac{2}{q-p}\sqrt{1+\frac{p-q}{x-p}} + C = \frac{2}{q-p}\sqrt{\frac{x-q}{x-p}} + \text{Const}$

c) Repeated integration by parts yeilds

$\int x^m(\ln x)^n\,dx = \left[\frac{x^{m+1}(\ln x)^n}{m+1}\right] - \int \frac{n}{m+1}\frac{x^{m+1}}{x}(\ln x)^{n-1}\,dx = \left[\frac{x^{m+1}(\ln x)^n}{m+1}\right] - \left[\frac{n}{(m+1)^2}x^{m+1}(\ln x)^{n-1}\right]$

$+ \int \frac{n(n-1)}{(m+1)^2}\frac{x^{m+1}}{x}(\ln x)^{n-2}\,dx \ldots$

$= x^{m+1}(\ln x)^n/(m+1) - n\,x^{m+1}(\ln x)^{n-1}/(m+1)^2 + n(n-1)\,x^{m+1}(\ln x)^{n-2}/(m+1)^3 - \ldots$

$+ (-1)^n n!\, x^{m+1}/(m+1)^{n+1} + \text{Const}$

$= x^{m+1}\left((\ln x)^n/(m+1) - n(\ln x)^{n-1}/(m+1)^2 + n(n-1)(\ln x)^{n-2}/(m+1)^3 - \ldots + (-1)^n n!/(m+1)^{n}\right.$

5. a) $(1-x^2)\,dy/dx = x(a-y)$ is seprable $\int \frac{dy}{a-y} = \int \frac{x\,dx}{1-x^2} = \frac{1}{2}\int \frac{1}{1-x} - \frac{1}{1+x}\,dx$

$\Rightarrow -\ln(a-y) = \frac{1}{2}\left(-\ln(1-x) - \ln(1+x)\right) + \text{Const} \Rightarrow a-y = C\sqrt{1-x^2}$

$\underline{y = a - C\sqrt{1-x^2}}$.

b) If $y = x\varphi(x)$ the resulting equation will be separable: $x^2(x\varphi' + \varphi) = x^2 + 3x^2\varphi + x^2\varphi^2 \Rightarrow x\varphi' = \varphi^2 + 2\varphi + 1 = (\varphi+1)^2 \Rightarrow \int \frac{d\varphi}{(\varphi+1)^2} = \int \frac{dx}{x}$

$\Rightarrow -1/(\varphi+1) = \ln x + C \Rightarrow x = Ae^{-\frac{1}{\varphi+1}} \Rightarrow \underline{x = Ae^{-x/(y+x)}}$

or $\underline{y = x/(C - \ln x) - x}$.

c) $y' + 2/x\,y = 1/x\,\sqrt{1+3x^2}$. IF is $e^{\int 2/x\,dx} = x^2$ we see $(x^2y)' = x\sqrt{1+3x^2}$

$\Rightarrow x^2y = (1+3x^2)^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{2} + C \Rightarrow y = \frac{C}{x^2} + \frac{1}{9}\frac{(1+3x^2)^{3/2}}{x^2}$

If $y(1) = 1$ then $1 = C + \frac{1}{9}4^{3/2} = C + 8/9$ & $C = 1/9$, $\underline{y = \frac{1}{9}\left(\frac{1}{x^2} + \frac{(1+3x^2)}{x^2}\right.}$

6. a) $y'' - 4y' + 3y = 8e^{-x} - 2e^{x}$,   ae: $\alpha^2 - 4\alpha + 3 = 0$ $(\alpha-3)(\alpha-1) = 0$

$\Rightarrow$ CF $y = Ae^{3x} + Be^{x}$. PI try $y = pe^{-x} + qxe^{x}$ & find

$(pe^{-x} + 2qe^{x} + qxe^{x}) - 4(-pe^{-x} + qe^{x} + qxe^{x}) + 3(pe^{-x} + qxe^{x}) =$

$= 8e^{-x} - 2e^{x}$

$\Rightarrow p(1+4+3) = 8 \Rightarrow p = 1$ & $2q - 4q = -2 \Rightarrow q = 1$

& $y = Ae^{3x} + Be^{x} + e^{-x} + xe^{x}$

$y(0) = 1 \Rightarrow 1 = A + B + 1 \Rightarrow A = -B$

$y'(0) = 0 \Rightarrow 0 = 3A + B - 1 + 1 \Rightarrow 3A + B = 0$

$\Big\}$ $A = B = 0$

$\underline{y = xe^{x} + e^{-x}}$

(Similar seen)

b) $x^2 y'' + xy' + y = \sin(\ln x)$. If $x = e^t$ this becomes

$\dfrac{d^2y}{dt^2} + y = \sin t \Rightarrow y = A\sin t + B\cos t + \alpha t\sin t + \beta t\cos t$

with $\alpha(-t\sin t + 2\cos t) + \beta(-t\cos t + 2(-\sin t)) + \alpha t\sin t + \beta t\cos t = \sin t$

$\Rightarrow \alpha = 0$ & $\beta = -\frac{1}{2}$

& $y = A\sin t + B\cos t - \frac{1}{2}t\cos t$, $y(x) = A\sin(\ln x) + B\cos(\ln x)$

$- \frac{1}{2}\ln x\cos(\ln x)$